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TRANSFORM TECHNIQUES FOR MULTI-DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a novel Integral transform approach is applied to second-order differential equations of multi-delays type with pre-specified history functions. The delayed transform relationship and polynomial form of history are constructed. The existence and uniqueness, uniform convergence of Picard iterations, and exponential stability results are proved. In addition, a resolvent kernel and convolution type solution have been obtained using the Laplace transformation. Finally, two mechanical vibration models with multi-delays are considered as examples.

Keywords. Linear multi-delay differential equations; General integral transforms; Stability theory of functional; Initial value problems, Existence and Uniqueness; Analytical theory of differential equations.
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1. Introduction

Delay differential equations (DDEs) arise in mathematical models in which the evolution of a system depends not only on its present state but also on its previous states. Such equations are encountered in biological systems, engineering processes, population dynamics, control theory, and mechanical systems. In many practical situations, the presence of more than one time delay is necessary to represent different feedback mechanisms, transmission times, or memory effects. Consequently, multi-delay differential equations provide a useful mathematical framework for describing systems with several delayed interactions. Recent studies have investigated delay differential equations in biological and mathematical applications, including infectious disease models and other systems with delayed dynamics [1, 11, 14, 16].

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The analytical and numerical treatment of delay differential equations has received considerable attention. In particular, integral transforms provide an effective means of converting differential equations into algebraic or functional equations in a transform domain. The Laplace transform is widely used for obtaining analytical representations of linear delay differential equations. For example, Sherman et al.[20] derived analytical solutions of linear delay-differential equations with Dirac delta function inputs using the Laplace transform. Agiza et al. [1] employed a Taylor series transform to investigate small two-delay differential equations arising in Parkinson's disease modelling. Similarly, Elfouly and Sohaly [11] considered a two-delay representation of the Van der Pol model. These investigations demonstrate the usefulness of transform-based techniques in analysing systems involving delayed terms.

Several alternative integral transforms have also been introduced or adapted for solving differential and fractional differential equations. Alderremy et al. [3] developed a new transform iterative method for solving certain Klein–Gordon equations. Benattia and Belghaba [9] applied the Aboodh transform to fractional delay differential equations, while Benattia et al.[8] investigated a new type of integral transform and its applications to differential equations. More recently, transform-based approaches have been used to study the stability of multi-delay systems. Veerabathiran et al. [23] examined a multiple-delay model using the Laplace transform. Veerabathiran et al. [22] investigated multi-delay differential equations through the Aboodh transform, whereas Veerabathiran et al. [24] considered an Elzaki-transform approach to the stability analysis of a multi-delay model. Although these studies establish the relevance of integral transforms to delay-system analysis, further investigation is useful for developing a unified treatment that incorporates prescribed history functions, solution representations, and theoretical properties for second-order multi-delay equations.Selvam et al. [19] investigated Caputo-Hadamard fractional differential equations involving delay and pantograph terms, contributing to the study of fractional-order systems with memory and delayed effects.Furthermore, Guerarra et al. [13] studied optimal ranks of nonlinear matrix expressions with applications, providing related developments in nonlinear matrix analysis and computational mathematics.

Fractional-order differential equations and fractional delay models have also attracted considerable interest because fractional operators can describe memory-dependent and hereditary behaviour. Ansari et al. [6] studied delay differential equations involving a piecewise Caputo fractional-order derivative. Benattia and Belghaba [9] investigated fractional delay differential equations using the Aboodh transform, while Tunç and Tunç [21] established Ulam-type stability results for fractional integro-delay differential and integral equations involving the ψ-Hilfer operator. Related developments in fractional mathematical modelling include the discrete fractional-order model considered by Batiha et al. [7]. In addition, Akrour and Bellour [2] investigated stability, periodicity, and oscillation properties for systems of difference equations. Although difference equations and fractional models are not identical to the second-order multi-delay problem considered here, these studies illustrate the broader interest in stability and qualitative properties of systems with memory or discrete and delayed dynamics.

Mehellou and Khirani [15] investigated numerical solutions of nonlinear quadratic Fredholm integral equations, while Aljazzazi et al. [4] established existence results for conformable evolution differential equations with non-local conditions; moreover, El Hassani and El Kahoui [10] studied exponential boundary stabilization of a one-dimensional anti-damped wave equation using an integral transformation.

Theoretical properties such as existence, uniqueness, stability, oscillation and controllability are essential in determining the reliability of mathematical models involving delays. Roy et al. [17] studied approximate controllability for second-order differential equations with delays in the input and state variables. Salah et al. [18] investigated oscillation criteria for second-order half-linear delay differential equations. These works address important qualitative properties of delay systems and motivate further theoretical analysis for different classes of multi-delay equations. In particular, establishing existence and uniqueness under suitable assumptions is necessary before deriving and interpreting analytical solution representations.

Multi-delay effects are also important in mechanical and structural vibration systems, where delayed feedback can influence the dynamic response and stability of a structure. Almutlg et al. [5] investigated bifurcation and chaotic dynamics in a cantilever beam subjected to triple time-delayed feedback control. Fang et al. [12] considered dynamics-guided constrained optimization of a piezoelectric nonlinear energy sink for structural vibration control. Zhang et al. [26] studied the mechanical properties and vibration control of bidirectional flexible-hinge-connected perovskite plates. Furthermore, Zhai et al.[25] developed an enhanced time-delay Smith predictor for a sliding-mode structural vibration controller based on an extended state observer. These investigations highlight the importance of delayed feedback and time-delay effects in mechanical and structural applications.

Despite the progress described above, there remains scope for a systematic analytical framework for second-order linear multi-delay differential equations with prescribed history functions. In particular, the treatment of delayed terms must account explicitly for the contribution of the history function over the corresponding delay intervals. Moreover, a solution representation obtained through an integral transform should be complemented by appropriate theoretical results concerning existence, uniqueness, convergence, and stability. These considerations motivate the present study.

In this paper, we consider the second-order multi-delay differential equation

$$\omega''(\rho) + \psi_1\omega(\rho) + \psi_2\omega(\rho-\tau_1) + \psi_3\omega(\rho-\tau_2) = f(\rho), \qquad \rho \geq 0, \tag{1.1}$$

subject to the initial conditions

$$\omega(0) = a_0, \qquad \omega'(0) = a_1, \tag{1.2}$$

and the prescribed history function

$$\omega(\rho) = \phi(\rho), \qquad -\tau_{\max} \leq \rho \leq 0, \qquad \tau_{\max} = \max\{\tau_1, \tau_2\}. \tag{1.3}$$

Here, ψ_1,ψ_2 and ψ_3. are real-valued system parameters, $\tau_1, \tau_2 > 0$ denote the time delays, and $f(\rho)$ is a given forcing function.

To investigate this problem, we develop a Integral Transform approach and establish a general transformation formula for delayed functions involving the prescribed history. In particular, an explicit expression is obtained for polynomial history functions of the form $\phi(\rho) = \rho^m$, where m is a non-negative integer. The main contributions of this study are summarized as follows:

- A Integral Transform framework is formulated for the considered second-order multi-delay differential equation. A general transform formula for delayed functions is derived by explicitly incorporating the prescribed history function. The corresponding expression for polynomial history functions is also obtained.
- Existence and uniqueness of solutions are investigated under suitable assumptions on the coefficients and the considered time interval. The convergence of Picard successive approximations is analysed, and a transform-domain solution representation is developed.
- Mechanical vibration examples with multiple delayed feedback terms are presented to illustrate the application of the proposed analytical framework.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries and fundamental properties of the Integral Transform. Section 3 presents the transform-based analysis of the multi-delay differential equation, including the delayed-function formula and the solution representation. The theoretical results concerning existence, uniqueness, convergence, and stability are discussed in the subsequent analysis. Section 4 presents mechanical vibration examples to demonstrate the applicability of the proposed approach. Finally, the main findings are summarized in the conclusion.

2. Preliminaries

Definition 2.1. Let $\omega(\rho)$ be a function of ρ specified for $\rho \geq 0$. Then the Laplace transform of $\omega(\rho)$,denoted by $\boldsymbol{F}[\omega(\rho)]$, is defined by(p is the transform variable)

$$\boldsymbol{F}[\omega(\rho)] = f(p) = \int_0^\infty \omega(\rho)\, e^{-p\rho} d\rho, \tag{2.1}$$

Definition 2.2. *Let the Integral transform denotes by the notation $\boldsymbol{N}(.)$ and a function f of exponential order is defined on a set. Then,the New transform off is given by the integral equation: (σ Transform parameter and γ a nonzero scaling parameter)*[8]

$$\boldsymbol{N}[\omega(\rho)] = \left(\frac{1}{\gamma^2}\right) \int_0^\infty \omega(\rho)\, e^{\left(\frac{\rho\sigma}{\gamma}\right)} d\rho, \quad Rel\left(\frac{\sigma}{\gamma}\right) < 0 \tag{2.2}$$

Theorem 2.3. *Let $\boldsymbol{N}[\omega(\rho)]$ be the Integral transform, then*

$$\boldsymbol{N}[d] = -\frac{d}{\sigma\gamma}, \frac{\sigma}{\gamma} > 0; \tag{2.3}$$

$$\boldsymbol{N}[\rho] = \frac{1}{\sigma^2}; \tag{2.4}$$

$$\boldsymbol{N}[\rho^n] = \frac{(-1)^{(n+1)} n! \gamma^{n-1}}{\sigma^{n+1}}, n \in N. \tag{2.5}$$

Theorem 2.4. *The Integral transform of an exponential function $e^{d\rho}$ is given by:*

$$\boldsymbol{N}[\omega(\rho)] = \left(\frac{1}{\gamma^2}\right) \int_0^\infty e^{d\rho}\, e^{\left(\frac{\rho\sigma}{\gamma}\right)} d\rho = -\frac{1}{\gamma(\sigma + d\gamma)} \tag{2.6}$$

Theorem 2.5. *The Integral transform of the trigonometric functions, such as* $\cos(d\rho)$ *and* $\sin(d\rho)$ *are defined by*

$$\boldsymbol{N}[\cos(d\rho)] = -\frac{\sigma}{\gamma(\sigma^2 + d^2\gamma^2)} \tag{2.7}$$

and

$$\boldsymbol{N}[\sin(d\rho)] = \frac{d}{(\sigma^2 + d^2\gamma^2)} \tag{2.8}$$

Inverse properties. The Inverse properties of the proposed Integral transform are essential for recovering the original-domain representation from its transformed form. In particular, the following relations are used throughout the derivation of the proposed solution procedure

$$\boldsymbol{N^{-1}}\Big[-\frac{d}{\sigma\gamma}\Big] = d, \frac{\sigma}{\gamma} > 0; \tag{2.9}$$

$$\boldsymbol{N^{-1}}[\frac{1}{\sigma^2}] = \rho; \tag{2.10}$$

$$\boldsymbol{N^{-1}}[\frac{(-1)^{(n+1)}\gamma^{n-1}}{\sigma^{n+1}}] = \frac{\rho^n}{n!}, n \in N. \tag{2.11}$$

Theorem 2.6. *The Integral transform of higher order derivatives.Let a function* $\omega(\rho)$ *be in set* $\boldsymbol{N}[\omega^{(n)}(\rho)]$ *its Integral transform, then*

$$\boldsymbol{N}[\omega^{(n)}(\rho)] = (-1)^n \Big(\frac{\sigma}{\gamma}\Big)^n \boldsymbol{N}[(\sigma,\gamma)] - \frac{1}{\gamma^2} \sum_{k=0}^{n-1} \Big(-\frac{\sigma}{\gamma}\Big)^{n-(k+1)} \omega^{(k)}(0), n \geq 1 \tag{2.12}$$

Theorem 2.7. *Let* $\tau \geq 0$, $\gamma > 0$, *and* $\sigma < 0$. *Let* $\omega : [-\tau, \infty) \to \mathbb{R}$ *be a piecewise continuous function of exponential order, and suppose that its history function is prescribed by*

$$\omega(\rho) = \phi(\rho), \qquad -\tau \leq \rho \leq 0.$$

Define the Integral Transform by equation (2.2) Then the transform of the delayed function $\omega(\rho - \tau)$ *is given by*

$$\boldsymbol{N}[\omega(\rho - \tau)] = e^{\frac{\sigma\tau}{\gamma}} \boldsymbol{N}[\omega(\rho)] + \frac{e^{\frac{\sigma\tau}{\gamma}}}{\gamma^2} \int_{-\tau}^{0} \phi(u) e^{\frac{\sigma u}{\gamma}}\, du.$$

Proof:
By the definition of the Integral Transform,

$$\boldsymbol{N}[\omega(\rho-\tau)] = \frac{1}{\gamma^2}\int_0^\infty \omega(\rho-\tau)e^{\frac{\sigma\rho}{\gamma}}\,d\rho.$$

Splitting the integral at $\rho=\tau$, we obtain

$$\begin{aligned}\boldsymbol{N}[\omega(\rho-\tau)] = &\frac{1}{\gamma^2}\int_0^\tau \phi(\rho-\tau)e^{\frac{\sigma\rho}{\gamma}}\,d\rho\\ &+\frac{1}{\gamma^2}\int_\tau^\infty \omega(\rho-\tau)e^{\frac{\sigma\rho}{\gamma}}\,d\rho.\end{aligned}$$

In the first integral, let $u=\rho-\tau$. Then $\rho=u+\tau$, and the limits become $-\tau$ and 0. Thus,

$$\frac{1}{\gamma^2}\int_0^\tau \phi(\rho-\tau)e^{\frac{\sigma\rho}{\gamma}}\,d\rho = \frac{e^{\frac{\sigma\tau}{\gamma}}}{\gamma^2}\int_{-\tau}^0 \phi(u)e^{\frac{\sigma u}{\gamma}}\,du.$$

Similarly, in the second integral, let $u=\rho-\tau$. Then $\rho=u+\tau$, and the limits become 0 and ∞. Therefore,

$$\frac{1}{\gamma^2}\int_\tau^\infty \omega(\rho-\tau)e^{\frac{\sigma\rho}{\gamma}}\,d\rho = \frac{e^{\frac{\sigma\tau}{\gamma}}}{\gamma^2}\int_0^\infty \omega(u)e^{\frac{\sigma u}{\gamma}}\,du = e^{\frac{\sigma\tau}{\gamma}}\boldsymbol{N}[\omega(\rho)].$$

Combining these two expressions, we obtain

$$\boldsymbol{N}[\omega(\rho-\tau)] = e^{\frac{\sigma\tau}{\gamma}}\boldsymbol{N}[\omega(\rho)] + \frac{e^{\frac{\sigma\tau}{\gamma}}}{\gamma^2}\int_{-\tau}^0 \phi(u)e^{\frac{\sigma u}{\gamma}}\,du.$$

Hence, the theorem is proved.

Corollary 2.8. *Let $m\in\mathbb{N}_0$, $\tau\geq 0$, $\gamma>0$, and $\sigma<0$. Suppose that the history function is given by*

$$\phi(\rho)=\rho^m, \qquad -\tau\leq\rho\leq 0. \tag{2.13}$$

Then the Integral Transform of the delayed function is

$$\boldsymbol{N}[\omega(\rho-\tau)] = e^{\frac{\sigma\tau}{\gamma}}\boldsymbol{N}[\omega(\rho)] + \frac{e^{\frac{\sigma\tau}{\gamma}}}{\gamma^2}\int_{-\tau}^0 \phi(u)e^{\frac{\sigma u}{\gamma}}\,du. \tag{2.14}$$

Moreover, the history integral can be evaluated explicitly as

$$\int_{-\tau}^0 u^m e^{\frac{\sigma u}{\gamma}}\,du = (-1)^m\frac{\gamma^{m+1}}{\sigma^{m+1}}\int_0^{\frac{\sigma\tau}{\gamma}} x^m e^{-x}\,dx. \tag{2.15}$$

Using the incomplete gamma integral, We obtain

$$\boldsymbol{N}[\omega(\rho-\tau)] = e^{\frac{\sigma\tau}{\gamma}}\boldsymbol{N}[\omega(\rho)] + \frac{(-1)^m m!\gamma^{m-1}}{\sigma^{m+1}}\left[e^{\frac{\sigma\tau}{\gamma}} - \sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\sigma\tau}{\gamma}\right)^k\right]. \tag{2.16}$$

Theorem 2.9. *Let $T>0$, $\tau_1,\tau_2>0$, and $\tau=\max\{\tau_1,\tau_2\}$. Consider the second-order multi-delay differential equation (1.1) subject to the history condition (1.3) and the initial conditions (1.2) Assume that $f\in C([0,T])$, $\phi\in C([-\tau,0])$, $\phi(0)=a_0$ Define $L=|\psi_1|+|\psi_2|+|\psi_3|$. If*

$$\frac{T^2}{2}\left(|\psi_1|+|\psi_2|+|\psi_3|\right)<1, \tag{2.17}$$

then the multi-delay differential equation (1.1)–(1.2) *admits a unique solution* $\omega \in C^2([0,T])$.

Proof:
Let

$$X = \{\omega \in C([0,T]) : \omega(0) = a_0\}.$$

Equip X with the supremum norm

$$\|\omega\|_\infty = \max_{0\le\rho\le T} |\omega(\rho)|. \tag{2.18}$$

Since X is a closed subset of the Banach space $C([0,T])$, the space X is complete. For every $\omega \in X$, define its extended history function by

$$\widetilde{\omega}(\rho) = \begin{cases} \phi(\rho), & -\tau \le \rho \le 0, \\ \omega(\rho), & 0 \le \rho \le T. \end{cases} \tag{2.19}$$

The condition $\phi(0) = a_0 = \omega(0)$ guarantees continuity at $\rho = 0$. Integrating equation (1.1) twice from 0 to ρ, and using

$$\omega(0) = a_0, \qquad \omega'(0) = a_1,$$

we obtain

$$\omega(\rho) = a_0 + a_1\rho + \int_0^\rho (\rho - s)\Big[f(s) - \psi_1\widetilde{\omega}(s) - \psi_2\widetilde{\omega}(s-\tau_1) - \psi_3\widetilde{\omega}(s-\tau_2)\Big]\, ds. \tag{2.20}$$

Define an operator

$$\mathcal{T} : X \longrightarrow X$$

by

$$(\mathcal{T}\omega)(\rho) = a_0 + a_1\rho + \int_0^\rho (\rho - s)\Big[f(s) - \psi_1\widetilde{\omega}(s) - \psi_2\widetilde{\omega}(s-\tau_1) - \psi_3\widetilde{\omega}(s-\tau_2)\Big]\, ds. \tag{2.21}$$

Since the integrand is continuous, $\mathcal{T}\omega$ is continuous on $[0,T]$. Moreover,

$$(\mathcal{T}\omega)(0) = a_0.$$

Hence,

$$\mathcal{T}\omega \in X.$$

Therefore, $\mathcal{T}$ maps X into itself. Let $\omega_1, \omega_2 \in X$. From (2.21), we obtain

$$\begin{aligned}(\mathcal{T}\omega_1)(\rho) - (\mathcal{T}\omega_2)(\rho) = & -\int_0^\rho (\rho - s)\Big[\psi_1(\widetilde{\omega}_1(s) - \widetilde{\omega}_2(s)) + \psi_2\big(\widetilde{\omega}_1(s-\tau_1) - \widetilde{\omega}_2(s-\tau_1)\big) \\ & + \psi_3\big(\widetilde{\omega}_1(s-\tau_2) - \widetilde{\omega}_2(s-\tau_2)\big)\Big]\, ds.\end{aligned} \tag{2.22}$$

Taking absolute values and applying the triangle inequality, we get

$$\begin{aligned}|(\mathcal{T}\omega_1)(\rho) - (\mathcal{T}\omega_2)(\rho) \le & \int_0^\rho (\rho - s)\Big[|\psi_1||\widetilde{\omega}_1(s) - \widetilde{\omega}_2(s)| \quad + |\psi_2||\widetilde{\omega}_1(s-\tau_1) - \widetilde{\omega}_2(s-\tau_1)| \\ & + |\psi_3||\widetilde{\omega}_1(s-\tau_2) - \widetilde{\omega}_2(s-\tau_2)|\Big]\, ds.\end{aligned} \tag{2.23}$$

Since both functions have the same prescribed history, we have

$$|\widetilde{\omega}_1(s-\tau_i) - \widetilde{\omega}_2(s-\tau_i)| \leq \|\omega_1 - \omega_2\|_\infty, \qquad i = 1, 2. \tag{2.24}$$

Similarly,

$$|\widetilde{\omega}_1(s) - \widetilde{\omega}_2(s)| \leq \|\omega_1 - \omega_2\|_\infty.$$

Substituting these inequalities into (2.23), we obtain

$$|(\mathcal{T}\omega_1)(\rho) - (\mathcal{T}\omega_2)(\rho)| \leq L\|\omega_1 - \omega_2\|_\infty \int_0^\rho (\rho - s)\, ds. \tag{2.25}$$

Evaluating the integral gives, Consequently,

$$|(\mathcal{T}\omega_1)(\rho) - (\mathcal{T}\omega_2)(\rho)| \leq \frac{L\rho^2}{2}\|\omega_1 - \omega_2\|_\infty. \tag{2.26}$$

Taking the supremum over $0 \leq \rho \leq T$, we obtain

$$\|\mathcal{T}\omega_1 - \mathcal{T}\omega_2\|_\infty \leq \frac{LT^2}{2}\|\omega_1 - \omega_2\|_\infty. \tag{2.27}$$

we can choose the contraction constant

$$q = \frac{LT^2}{2}, \qquad 0 \leq q < 1.$$

Thus, equation (2.27) becomes

$$\|\mathcal{T}\omega_1 - \mathcal{T}\omega_2\|_\infty \leq q\|\omega_1 - \omega_2\|_\infty. \tag{2.28}$$

Therefore, $\mathcal{T}$ is a contraction mapping on the complete metric space X. By the Banach fixed-point theorem, there exists a unique $\omega \in X$ such that

$$\mathcal{T}\omega = \omega. \tag{2.29}$$

Hence, the integral equation (2.20) admits a unique continuous solution. This completes the proof.

Theorem 2.10. *Let $T > 0$, $\tau_1, \tau_2 > 0$, and $\tau = \max\{\tau_1, \tau_2\}$. Consider the second-order multi-delay differential equation (1.1) subject to the history condition (1.3) and the initial conditions (1.2) Assume that $f \in C([0, T])$, $\phi \in C([-\tau, 0])$, and $\phi(0) = a_0$,*

$$L = |\psi_1| + |\psi_2| + |\psi_3|.$$

Define the Picard successive approximations by

$$\omega_0(\rho) = a_0 + a_1\rho, \tag{2.30}$$

and

$$\omega_{n+1}(\rho) = a_0 + a_1\rho + \int_0^\rho (\rho - s)\Big[f(s) - \psi_1\widetilde{\omega}_n(s) - \psi_2\widetilde{\omega}_n(s-\tau_1) - \psi_3\widetilde{\omega}_n(s-\tau_2)\Big]\, ds, \tag{2.31}$$

where

$$\widetilde{\omega}_n(s) = \begin{cases} \phi(s), & -\tau \leq s \leq 0, \\ \omega_n(s), & 0 < s \leq T. \end{cases}$$

Then the sequence $\{\omega_n\}_{n=0}^{\infty}$ *converges uniformly on* $[0,T]$ *to the unique solution* $\omega \in C^2([0,T])$ *of* (1.1)–(1.2). *Moreover, for every* $n \geq 0$, *the successive approximations satisfy*

$$\|\omega_{n+1} - \omega_n\|_\infty \leq \frac{M(LT^2)^n}{n!}, \tag{2.32}$$

where

$$M = \|\omega_1 - \omega_0\|_\infty .$$

Consequently, the error satisfies

$$\|\omega - \omega_n\|_\infty \leq M \sum_{k=n}^{\infty} \frac{(LT^2)^k}{k!}. \tag{2.33}$$

Proof:
Integrating equation (1.1) twice with respect to ρ and using the initial conditions, we obtain

$$\omega(\rho) = a_0 + a_1\rho + \int_0^\rho (\rho - s)\Big[f(s) - \psi_1\omega(s) - \psi_2\omega(s-\tau_1) - \psi_3\omega(s-\tau_2)\Big]\, ds. \tag{2.34}$$

This integral equation motivates the Picard iteration defined in (2.31). Define

$$e_n(\rho) = \omega_n(\rho) - \omega_{n-1}(\rho), \qquad n \geq 1.$$

Also, define

$$\widetilde{e}_n(s) = \begin{cases} 0, & -\tau \leq s \leq 0, \\ e_n(s), & 0 < s \leq T. \end{cases}$$

Subtracting two consecutive Picard iterations gives

$$e_{n+1}(\rho) = -\int_0^\rho (\rho - s)\Big[\psi_1\widetilde{e}_n(s) + \psi_2\widetilde{e}_n(s-\tau_1) + \psi_3\widetilde{e}_n(s-\tau_2)\Big]\, ds. \tag{2.35}$$

Taking absolute values and using $0 \leq \rho - s \leq T$, we obtain

$$|e_{n+1}(\rho)| \leq T\int_0^\rho \Big[|\psi_1||\widetilde{e}_n(s)| + |\psi_2||\widetilde{e}_n(s-\tau_1)| + |\psi_3||\widetilde{e}_n(s-\tau_2)|\Big]\, ds. \tag{2.36}$$

Let

$$D_n(\rho) = \sup_{0\leq u\leq \rho} |e_n(u)|.$$

Since the extended difference is zero for negative arguments, we have

$$|\widetilde{e}_n(s-\tau_i)| \leq D_n(s), \qquad i = 1, 2.$$

Therefore, equation (2.36) implies

$$D_{n+1}(\rho) \leq LT\int_0^\rho D_n(s)\, ds. \tag{2.37}$$

By definition,

$$D_1(\rho) \leq M.$$

Applying (2.37) repeatedly, we obtain

$$D_2(\rho) \leq MLT\rho,$$

and

$$D_3(\rho) \leq \frac{M(LT)^2\rho^2}{2!}.$$

Continuing this process, mathematical induction yields

$$D_n(\rho) \leq \frac{M(LT)^{n-1}\rho^{n-1}}{(n-1)!}. \tag{2.38}$$

Consequently, setting $\rho = T$ gives

$$\|\omega_{n+1} - \omega_n\|_\infty \leq \frac{M(LT^2)^n}{n!}. \tag{2.39}$$

For integers $m > n$, we have

$$\omega_m(\rho) - \omega_n(\rho) = \sum_{k=n}^{m-1} \big(\omega_{k+1}(\rho) - \omega_k(\rho)\big).$$

Taking the supremum norm and applying (2.39), we obtain

$$\begin{aligned} \|\omega_m - \omega_n\|_\infty &\leq M \sum_{k=n}^{m-1} \frac{(LT^2)^k}{k!} \\ &\leq M \sum_{k=n}^{\infty} \frac{(LT^2)^k}{k!}. \end{aligned} \tag{2.40}$$

Since the exponential series

$$\sum_{k=0}^{\infty} \frac{(LT^2)^k}{k!} = e^{LT^2}$$

converges for every finite T, the right-hand side of (2.40) tends to zero as $n \to \infty$. Hence, $\{\omega_n\}$ is a uniformly Cauchy sequence in the Banach space $C([0,T])$. Therefore, there exists a continuous function $\omega \in C([0,T])$ such that

$$\lim_{n\to\infty} \|\omega_n - \omega\|_\infty = 0.$$

Thus, the Picard sequence converges uniformly on $[0,T]$.

Theorem 2.11. *Consider the homogeneous second-order multi-delay differential equation (1.1) subject to the initial-history conditions (1.3) and initial condtion (1.2) where $\tau = \max\{\tau_1, \tau_2\}, \tau_1, \tau_2 > 0$. the initial-history function $\phi \in C^1([-\tau, 0])$ satisfies $\phi(0) = a_0$ and $\phi'(0) = a_1$. The characteristic equation associated with (1.1) is*

$$\Delta(\lambda) = \lambda^2 + \psi_1 + \psi_2 e^{-\lambda\tau_1} + \psi_3 e^{-\lambda\tau_2} = 0. \tag{2.41}$$

If every characteristic root λ_j satisfies

$$\mathrm{Re}(\lambda_j) < 0, \qquad \text{for all } j, \tag{2.42}$$

then the zero solution of (1.1) is exponentially stable. More precisely, there exist constants $K > 0$ and $\delta > 0$ such that

$$|\omega(\rho)| + |\omega'(\rho)| \leq Ke^{-\delta\rho}\|\phi\|_{C^1([-\tau,0])}, \qquad \rho \geq 0. \tag{2.43}$$

Furthermore, if at least one characteristic root satisfies

$$\mathrm{Re}(\lambda_j) > 0, \tag{2.44}$$

then the zero solution is unstable.

Proof:
For the stability investigation, we consider the homogeneous form of the given multi-delay differential equation by taking $f(\rho) = 0$. The stability of the zero solution is determined by the behaviour of solutions corresponding to small initial-history perturbations. Assume a trial solution of the form

$$\omega(\rho) = e^{\lambda\rho}, \tag{2.45}$$

where $\lambda \in \mathbb{C}$ is a characteristic root. Differentiating, we obtain

$$\omega''(\rho) = \lambda^2 e^{\lambda\rho}. \tag{2.46}$$

For the delayed terms,

$$\omega(\rho - \tau_1) = e^{\lambda\rho}e^{-\lambda\tau_1}, \tag{2.47}$$

and

$$\omega(\rho - \tau_2) = e^{\lambda\rho}e^{-\lambda\tau_2}. \tag{2.48}$$

Substituting these expressions into (1.1), we obtain

$$e^{\lambda\rho}\left[\lambda^2 + \psi_1 + \psi_2 e^{-\lambda\tau_1} + \psi_3 e^{-\lambda\tau_2}\right] = 0. \tag{2.49}$$

Since $e^{\lambda\rho} \neq 0$, the characteristic equation is

$$\Delta(\lambda) = \lambda^2 + \psi_1 + \psi_2 e^{-\lambda\tau_1} + \psi_3 e^{-\lambda\tau_2} = 0. \tag{2.50}$$

Let λ_j denote the roots of (2.50). Suppose that all characteristic roots satisfy

$$\mathrm{Re}(\lambda_j) < 0. \tag{2.51}$$

We show that the roots are uniformly separated from the imaginary axis. Indeed, for every root with $\mathrm{Re}(\lambda) \geq -1$, the characteristic equation gives

$$|\lambda|^2 \leq |\psi_1| + |\psi_2|e^{\tau_1} + |\psi_3|e^{\tau_2}. \tag{2.52}$$

Consequently, all characteristic roots in the half-plane $\mathrm{Re}(\lambda) \geq -1$ lie in a bounded region of the complex plane. Since the characteristic function is analytic and is not identically zero, it has only finitely many zeros in this bounded region. Under the assumed condition that every root has negative real part, there exists $\delta > 0$ such that

$$\mathrm{Re}(\lambda_j) \leq -\delta \qquad \text{for all } j. \tag{2.53}$$

By the spectral stability theorem for linear retarded differential equations with constant coefficients, the solution operator satisfies an exponential decay estimate whenever all characteristic roots lie strictly in the left half-plane. Therefore, there exist constants $K > 0$ and $\delta > 0$ such that

$$|\omega(\rho)| + |\omega'(\rho)| \leq Ke^{-\delta\rho}\|\phi\|_{C^1([-\tau,0])}, \qquad \rho \geq 0. \tag{2.54}$$

Taking the limit as $\rho \to \infty$, we obtain

$$\lim_{\rho\to\infty} \omega(\rho) = 0, \tag{2.55}$$

and

$$\lim_{\rho\to\infty} \omega'(\rho) = 0. \tag{2.56}$$

Hence, the zero solution is exponentially stable and therefore asymptotically stable. Now suppose that at least one characteristic root λ_0 satisfies

$$\mathrm{Re}(\lambda_0) > 0. \tag{2.57}$$

From the characteristic equation, the function

$$\omega(\rho) = e^{\lambda_0\rho} \tag{2.58}$$

is a nontrivial complex solution of the homogeneous equation. Since

$$|e^{\lambda_0\rho}| = e^{\mathrm{Re}(\lambda_0)\rho}, \tag{2.59}$$

the corresponding solution grows exponentially as $\rho \to \infty$. Because the coefficients are real, the real and imaginary parts of this complex solution are real-valued solutions. At least one of them is unbounded. Therefore, arbitrarily small initial-history perturbations can produce solutions that grow without bound. Consequently, the zero solution is unstable. Thus, the theorem is proved.

Theorem 2.12. *Consider multy delay differential equation (1.1) where $n \in \mathbb{N}$, $\tau_1, \tau_2 > 0$ are two distinct delay parameters, and $f(\rho)$ is a given continuous forcing function. for $k = 0, 1$. Then, under the Integral Transform $\boldsymbol{N}$, the transformed solution is*

$$\begin{aligned}\omega(\rho) = &\int_0^\rho f(s)K(\rho-s)\,ds + a_0K'(\rho) + a_1K(\rho)\\ &- \psi_2\int_0^\rho a_m(s,\tau_1)K(\rho-s)\,ds - \psi_3\int_0^\rho a_m(s,\tau_2)K(\rho-s)\,ds.\end{aligned}$$

Proof:
Applying the Integral transform $\boldsymbol{N}$ to both sides of (1.1), we obtain

$$\boldsymbol{N}[\omega^{(n)}(\rho)] + \psi_1\boldsymbol{N}[\omega(\rho)] + \psi_2\boldsymbol{N}[\omega(\rho-\tau_1)] + \psi_3\boldsymbol{N}[\omega(\rho-\tau_2)] = \boldsymbol{N}[f(\rho)]. \tag{2.60}$$

Collecting the terms containing $\boldsymbol{N}[\omega]$, and use Corollary (2.8) we have

$$\begin{aligned}\left[\left(\frac{\sigma}{\gamma}\right)^2 + \psi_1 + \psi_2 e^{\frac{\sigma\tau_1}{\gamma}} + \psi_3 e^{\frac{\sigma\tau_2}{\gamma}}\right]\boldsymbol{N}[\omega] = \boldsymbol{N}[f] + \frac{1}{\gamma^2}\Big(-\frac{\sigma}{\gamma}\omega(0) + \omega'(0)\Big)\\ - \Big(\frac{\psi_2(-1)^m m!\gamma^{m-1}}{\sigma^{m+1}}\left[1 - e^{-\frac{\sigma\tau_1}{\gamma}}\sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\sigma\tau_1}{\gamma}\right)^k\right]\Big)\\ - \Big(\frac{\psi_3(-1)^m m!\gamma^{m-1}}{\sigma^{m+1}}\left[1 - e^{-\frac{\sigma\tau_2}{\gamma}}\sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\sigma\tau_2}{\gamma}\right)^k\right]\Big)\end{aligned} \tag{2.61}$$

Define

$$Q_m(\tau_i,\sigma,\gamma) = \frac{(-1)^m m!\gamma^{m-1}}{\sigma^{m+1}}\left[1 - e^{-\frac{\sigma\tau_i}{\gamma}}\sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\sigma\tau_i}{\gamma}\right)^k\right] \qquad i = 1, 2. \tag{2.62}$$

Then, whenever the denominator below is nonzero, the transformed solution satisfies

$$\begin{aligned}\boldsymbol{N}[\omega(\rho)] = \frac{1}{\left(\frac{\sigma}{\gamma}\right)^2 + \psi_1 + \psi_2 e^{\frac{\sigma\tau_1}{\gamma}} + \psi_3 e^{\frac{\sigma\tau_2}{\gamma}}} \\ \times \left[\boldsymbol{N}[f(\rho)] + \frac{1}{\gamma^2}\Big(- \frac{\sigma}{\gamma}\omega(0) + \omega'(0)\Big) \right. \\ \left. - \psi_2 Q_m(\tau_1, \sigma, \gamma) - \psi_3 Q_m(\tau_2, \sigma, \gamma)\right].\end{aligned} \tag{2.63}$$

Consequently, the solution is represented formally by

$$\omega(\rho) = \boldsymbol{N}^{-1}\left[\frac{\boldsymbol{N}[f(\rho)] + \frac{1}{\gamma^2}\Big(- \frac{\sigma}{\gamma}\omega(0) + \omega'(0)\Big) - \psi_2 Q_m(\tau_1, \sigma, \gamma) - \psi_3 Q_m(\tau_2, \sigma, \gamma)}{\left(\frac{\sigma}{\gamma}\right)^2 + \psi_1 + \psi_2 e^{\frac{\sigma\tau_1}{\gamma}} + \psi_3 e^{\frac{\sigma\tau_2}{\gamma}}}\right]. \tag{2.64}$$

Provided that

$$\left(\frac{\sigma}{\gamma}\right)^2 + \psi_1 + \psi_2 e^{\frac{\sigma\tau_1}{\gamma}} + \psi_3 e^{\frac{\sigma\tau_2}{\gamma}} \neq 0, \tag{2.65}$$

Hence, the required transformed representation of the multi-delay differential equation is obtained. Introduce a new transform variable To simplify the inverse transform, put

$$p = -\frac{\sigma}{\gamma}, \qquad p > 0$$

Using the Laplace transform notation

$$W(p) = \int_0^\infty \omega(\rho) e^{-p\rho}\, d\rho,$$

we have

$$\boldsymbol{N}[\omega(\rho)] = \frac{W(p)}{\gamma^2}.$$

Similarly,

$$\boldsymbol{N}[f(\rho)] = \frac{F(p)}{\gamma^2},$$

where

$$F(p) = \int_0^\infty f(\rho) e^{-p\rho}\, d\rho.$$

Applying the Integral Transform to the governing equation and using the derivative property, we obtain

$$D(p)W(p) = F(p) + pa_0 + a_1 - \psi_2 A_m(p, \tau_1) - \psi_3 A_m(p, \tau_2),$$

where

$$A_m(p, \tau_i) = \frac{m!}{p^{m+1}}\left[\sum_{r=0}^{m} \frac{(-p\tau_i)^r}{r!} - e^{-p\tau_i}\right].$$

Consequently,

$$W(p) = \frac{F(p) + pa_0 + a_1 - \psi_2 A_m(p,\tau_1) - \psi_3 A_m(p,\tau_2)}{D(p)}.$$

First expand the denominator:

$$\frac{1}{D(p)} = \frac{1}{p^2}\frac{1}{1 + \dfrac{\psi_1 + \psi_2 e^{-p\tau_1} + \psi_3 e^{-p\tau_2}}{p^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{p^{2n+2}} \left(\psi_1 + \psi_2 e^{-p\tau_1} + \psi_3 e^{-p\tau_2}\right)^n.$$

Using the multinomial theorem,

$$\frac{1}{D(p)} = \sum_{n=0}^{\infty} \sum_{\substack{j,k,l\geq 0 \\ j+k+l=n}} \frac{(-1)^n n!}{j!k!l!} \times \frac{\psi_1^j \psi_2^k \psi_3^l e^{-p(k\tau_1 + l\tau_2)}}{p^{2n+2}}.$$

Applying the inverse Laplace transform and using

$$\mathcal{L}^{-1}\left[\frac{e^{-pd}}{p^q}\right] = \frac{(\rho - d)^{q-1}}{\Gamma(q)} H(\rho - d),$$

we obtain

$$K(\rho) = \sum_{n=0}^{\infty} \sum_{\substack{j,k,l\geq 0 \\ j+k+l=n}} \frac{(-1)^n n!}{j!k!l!} \psi_1^j \psi_2^k \psi_3^l \times \frac{(\rho - k\tau_1 - l\tau_2)^{2n+1}}{(2n+1)!} H(\rho - k\tau_1 - l\tau_2).$$

Next, by the convolution theorem,

$$\begin{aligned}\omega(\rho) = {} & \mathcal{L}^{-1}[F(p)] * K(\rho) + a_0 \mathcal{L}^{-1}[p/D(p)] + a_1 \mathcal{L}^{-1}[1/D(p)] \\ & - \psi_2 \mathcal{L}^{-1}\left[\frac{A_m(p,\tau_1)}{D(p)}\right] - \psi_3 \mathcal{L}^{-1}\left[\frac{A_m(p,\tau_2)}{D(p)}\right].\end{aligned}$$

Since $K(0) = 0$, the initial-condition contributions are $\mathcal{L}^{-1}\left[\frac{p}{D(p)}\right] = K'(\rho)$, and $\mathcal{L}^{-1}\left[\frac{1}{D(p)}\right] = K(\rho)$. Furthermore,

$$\mathcal{L}^{-1}[A_m(p,\tau_i)] = m! \sum_{r=0}^{m} \frac{(-\tau_i)^r \rho^{m-r}}{r!(m-r)!} - (\rho - \tau_i)^m H(\rho - \tau_i) = a_m(\rho, \tau_i).$$

Therefore, applying the convolution theorem to each remaining term, we obtain solution

$$\begin{aligned}\omega(\rho) = {} & \int_0^{\rho} f(s) K(\rho - s)\, ds + a_0 K'(\rho) + a_1 K(\rho) \\ & - \psi_2 \int_0^{\rho} a_m(s,\tau_1) K(\rho - s)\, ds - \psi_3 \int_0^{\rho} a_m(s,\tau_2) K(\rho - s)\, ds.\end{aligned}$$

3. Application to Mechanical Vibration Control with Multiple Delayed Feedback Signals

3.1. **Mathematical Model.** Mechanical vibration control is an important problem in engineering applications involving flexible structures, machine foundations, suspension systems, and rotating machinery. In delayed-feedback control systems, the current motion of a mechanical system depends not only on its present state but also on its displacement at previous times. Such delayed feedback may significantly influence the vibration response and the stability of the system.

Consider the following normalized second-order multi-delay differential equation (1.1) where $\omega(\rho)$ denotes the displacement response, ψ_1 is the instantaneous restoring coefficient, ψ_2 and ψ_3 are the delayed-feedback coefficients, and $\tau_1, \tau_2 > 0$ are the corresponding time delays. The function $f(\rho)$ represents an external excitation.

The prescribed history function is denoted by equation (1.3) The initial conditions is (1.2) A physical model with mass M can be obtained by dividing the governing equation by M, with the coefficients interpreted according to the selected mechanical system and feedback controller. Since no explicit velocity-dependent damping term is included, the present model focuses on the effect of delayed displacement feedback. The solution is represented formally by (2.64)

3.2. **Case 1: Two Positive Delayed-Feedback Coefficients.** Consider the parameter values $\psi_1 = 1, \psi_2 = 2, \psi_3 = 1, m = 1$ $\tau_1 = 0.5, \tau_2 = 0.3$ with the sinusoidal excitation $f(\rho) = \sin(\rho)$. Figure (1) presents the prescribed history and the numerical displacement response. The response exhibits oscillatory behavior with a time-varying amplitude. The two delayed signals and the instantaneous restoring term jointly affect the response. The signs of the feedback coefficients alone are not sufficient to establish stability, since the delays also affect the phase and magnitude of the feedback signals.

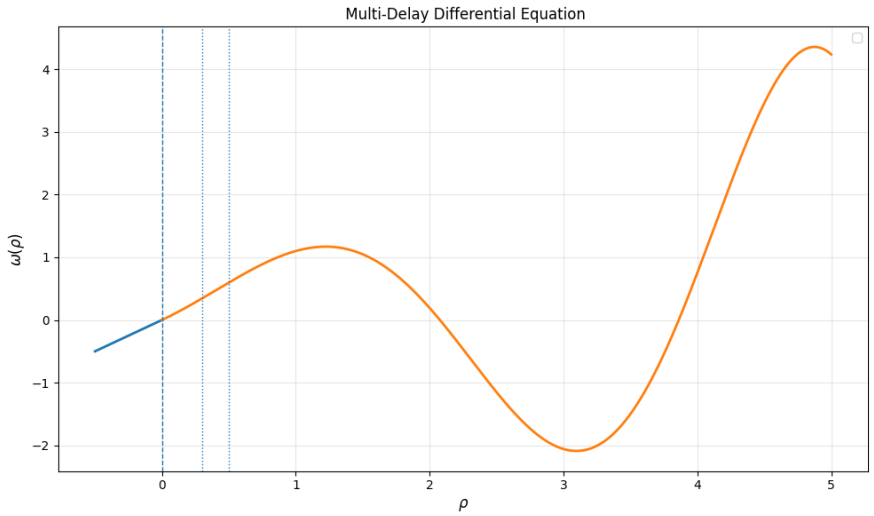

Fig. 1. Multi-delay differential equation

$$\omega(\rho) = \boldsymbol{N}^{-1}\left[\frac{\dfrac{1}{\sigma^2+\gamma^2} + \dfrac{1}{\gamma^2} - 2e^{\frac{0.5\sigma}{\gamma}}Q_1(0.5,\sigma,\gamma) - e^{\frac{0.3\sigma}{\gamma}}Q_1(0.3,\sigma,\gamma)}{\left(\dfrac{\sigma}{\gamma}\right)^2 + 1 + 2e^{\frac{0.5\sigma}{\gamma}} + e^{\frac{0.3\sigma}{\gamma}}}\right]. \tag{3.1}$$

Put

$$p = -\frac{\sigma}{\gamma}$$

Substituting all these expressions, we obtain

$$W(p) = \frac{\dfrac{1}{p^2+1} + 1 - \dfrac{2(1-0.5p-e^{-0.5p})}{p^2} - \dfrac{1-0.3p-e^{-0.3p}}{p^2}}{p^2+1+2e^{-0.5p}+e^{-0.3p}} \tag{3.2}$$

Expand the denominator, Using the multinomial theorem,

$$\frac{1}{D(p)} = \sum_{n=0}^{\infty} \sum_{\substack{j,k,l\geq 0 \\ j+k+l=n}} \frac{(-1)^n n!}{j!k!l!} \times \frac{2^k e^{-p(0.5k+0.3l)}}{p^{2n+2}}. \tag{3.3}$$

Apply the inverse Laplace transform

$$K(\rho) = \sum_{n=0}^{\infty} \sum_{\substack{j,k,l\geq 0 \\ j+k+l=n}} \frac{(-1)^n n!}{j!k!l!}\, 2^k \times \frac{(\rho-0.5k-0.3l)^{2n+1}}{(2n+1)!} H(\rho-0.5k-0.3l). \tag{3.4}$$

first delay,

$$a_1(\rho,0.5) = \begin{cases} \rho-0.5, & 0 \leq \rho < 0.5, \\ 0, & \rho \geq 0.5. \end{cases} \tag{3.5}$$

second delay,

$$a_1(\rho,0.3) = \begin{cases} \rho-0.3, & 0 \leq \rho < 0.3, \\ 0, & \rho \geq 0.3. \end{cases} \tag{3.6}$$

Apply the convolution theorem, we get the solution of the function

$$\begin{aligned} \omega(\rho) = K(\rho) &+ \int_0^{\rho} \sin(s) K(\rho-s)\, ds + 2\int_0^{\min(\rho,0.5)} (0.5-s) K(\rho-s)\, ds \\ &+ \int_0^{\min(\rho,0.3)} (0.3-s) K(\rho-s)\, ds. \end{aligned} \tag{3.7}$$

3.3. **Case 2: Mixed-Sign Delayed Feedback.** In the second case, choose $\psi_1 = 0$,$\psi_2 = 1$,$\psi_3 = -3$,$m = 2$, $\tau_1 = 0.3$,$\tau_2 = 0.8$ with $f(\rho) = \sin(\rho) + \rho^2$ Figure (2) shows the history and numerical response for this parameter set. The delayed signals have opposite signs and different delays. The negative coefficient of the second delayed term changes the influence of the corresponding past displacement on the current acceleration. The numerical response grows substantially over the plotted interval. This observation indicates that the chosen parameters do not

yield a bounded response on that interval; a formal stability conclusion would require additional analysis.

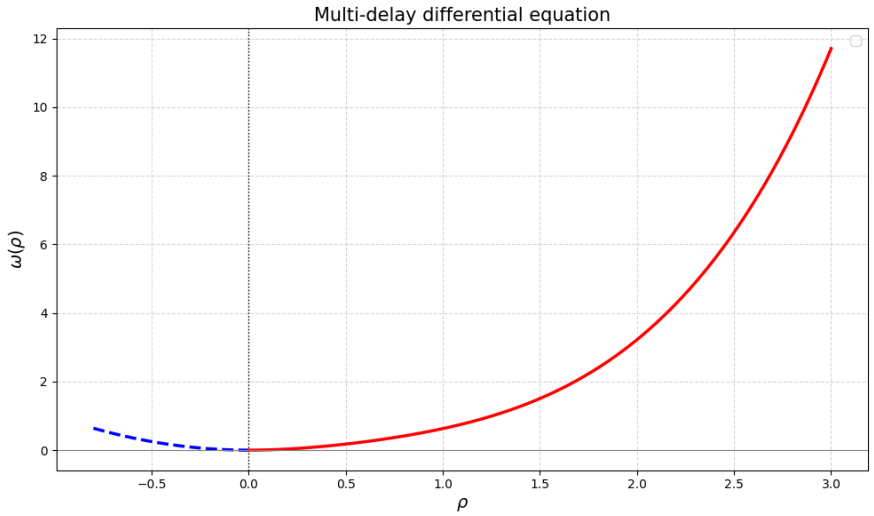

Fig. 2. Multi-delay differential equation

$$\omega(\rho) = \boldsymbol{N}^{-1}\left[\frac{\frac{1}{\sigma^2+\gamma^2} + \frac{-2\gamma}{\sigma^3} - e^{\frac{0.3\sigma}{\gamma}}Q_1(0.3,\sigma,\gamma) + 3e^{\frac{0.8\sigma}{\gamma}}Q_1(0.8,\sigma,\gamma)}{\left(\frac{\sigma}{\gamma}\right)^2 + e^{\frac{0.3\sigma}{\gamma}} - 3e^{\frac{0.8\sigma}{\gamma}}}\right]. \tag{3.8}$$

Put $p = -\frac{\sigma}{\gamma}$. Convert to Laplace transform

$$W(p) = \frac{\frac{1}{p^2+1} + \frac{1.83}{p} - \frac{4.2}{p^2} + \frac{6}{p^3} + \frac{2e^{-0.3s}}{p^3} - \frac{6e^{-0.8s}}{p^3}}{p^2 + e^{-0.3p} - 3e^{-0.8p}} \tag{3.9}$$

Expand the denominator using the geometric series Simplifying the coefficients,

$$\frac{1}{D(p)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+k}3^k}{p^{2n+2}} \times e^{-p(0.3n+0.5k)}. \tag{3.10}$$

Takve inverese Transform Therefore, the resolvent kernel is

$$K(\rho) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}3^k \times \frac{(\rho-0.3n-0.5k)^{2n+1}}{(2n+1)!} \times H(\rho-0.3n-0.5k). \tag{3.11}$$

First and second delay,

$$a_2(\rho,0.3) = \begin{cases}(0.3-\rho)^2, & 0 \le \rho < 0.3,\\ 0, & \rho \ge 0.3.\end{cases}$$

$$a_2(\rho, 0.8) = \begin{cases} (0.8-\rho)^2, & 0 \le \rho < 0.8, \\ 0, & \rho \ge 0.8. \end{cases}$$

This is the final convolution representation for your two-delay differential equation using the Integral Transform method.

$$\begin{aligned} \omega(\rho) = & \int_0^{\rho} (\sin s + s^2) K(\rho - s)\, ds - \int_0^{\min(\rho, 0.3)} (0.3 - s)^2 K(\rho - s)\, ds \\ & + 3 \int_0^{\min(\rho, 0.8)} (0.8 - s)^2 K(\rho - s)\, ds, \end{aligned} \tag{3.12}$$

4. Conclusion

In this work, using the Integral Transform is introduced to solve the second-order delay differential equations with specific initial functions. A generalized transformation relation for the delayed terms with polynomial initial functions has been derived. In addition, existence, uniqueness, convergence, and stability properties have been analyzed under appropriate assumptions. Also, a representation of the solution through resolvent kernel has been presented to represent the solution in terms of convolution integral form. At last, examples related to mechanical vibrations with multiple delayed feedbacks are given to show the applicability of the introduced framework. graphystyleamsplain

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